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STATISTICAL DISTRIBUTION OF THE BREAKING STRENGTH OF A BUNDLE OF CLASSICAL FIBERS

by Burt M. Rosenbaum

Lewis Research Center
Cleveland, Ohio

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ABSTRACT

It is shown that, for ideal bundles having a given number of fibers that will break under a load per fiber l_1, and a given number of fibers that will not break under a load l_2 $(>l_1)$, the average number of unbroken fibers in the bundle at loads between l_1 and l_2 depends linearly on the probability that an individual fiber picked at random will support a load l. Bundles are grouped according to the number of fibers in the bundle that can support that load per fiber at which the average total load supported by the bundle is a maximum. An approximation to the maximum of the average load supported by the bundle is arrived at for each group, and this maximum averaged over the groups constitutes a lower bound for the average breaking strength of the bundle. The results agree with those of Pierce and Daniels.

INTRODUCTION

A classical fiber is one that will support a load less than its breaking strength indefinitely without stretching or breaking but will break immediately under any load equal to or in excess of its breaking strength. The fundamental papers dealing with the breaking strengths of an ideal bundle (no friction or twist) of equal-length classical fibers are those by Pierce (ref. 1) and Daniels (ref. 2). Pierce was the first to obtain an approximation to the average breaking strength of a bundle of N such fibers. Daniels rigorously showed by sophisticated mathematical methods that the expression of Pierce is the correct asymptotic limit of the breaking strength as N gets large. He also found the asymptotic limit of the variance of the breaking strength for large N. In this report, the method of reasoning employed leads to the establishment of the aforementioned results from a viewpoint that is based on physical rather than mathematical insight.

Concept of Path

Let the distribution of breaking loads of a classical fiber be given by the frequency function $b(l)$, where $b(l)dl$ is the probability that a randomly selected individual fiber will have a breaking strength lying in dl at l. The cumulative distribution function

$$B(l) = \int_0^l b(\theta)d\theta$$

is the probability that a randomly selected fiber will break at some load less than l. Let $R(l) = 1 - B(l)$ denote the probability that a fiber will remain unbroken up to load l.

When a load L is hung on an ideal bundle of n classical fibers, the load will distribute itself evenly over the n fibers so that each fiber supports a load $l = L/n$. The breaking strength of a given fiber of the bundle does not depend on the action of the other fibers of the bundle and, hence, the breaking strength of the bundle may be determined from the strengths of the individual fibers. In particular, suppose one considers the bundle as separated into its independent individual fibers with separate loads l of equal magnitude on each fiber. Let this load l be allowed to increase monotonically from zero to the breaking strength of the strongest fiber. Then, at any point in the process where the individual fibers support a load l, the total load $L(l)$ supported by the bundle is the product of the number of unbroken fibers $n(l)$ and the load l acting on each unbroken fiber. Thus, the breaking strength of the bundle is the maximum of the total load L. After this maximum L has been reached, further increases in l cause a reduction in the total load $L(l) = ln(l)$. (If the fibers were not separated in this manner and if breaking strength were determined by increasing L until breakage occurred, no decrease in total load L would appear, and the bundle would immediately break after the breaking strength of the bundle had been reached.)

The situation may be depicted by means of figure 1. In figure 1(a), a typical plot of l against $R(l)$ is given. It can be seen that as l increases from zero to high values, $R(l)$ decreases from unity to zero or, if $R(l)$ is to be considered the independent variable, l increases from zero to high values as $R(l)$ decreases from unity to zero. The path followed by a bundle of N fibers averaged over the population of bundles is shown as the straight line in figure 1(b), where the expected number of surviving fibers is given by $E(n) = NR(l)$. The path taken by a typical bundle is shown in figure 1(c), where n for that bundle is plotted against $R(l)$. For large N at least, the actual path followed by a particular bundle would be expected to lie fairly close to the average path as given by the $E(n)$ line.

Results of Pierce and Daniels

The average load supported by the bundle as a function of l is

$$\bar{L}(l) = lE(n) = NlR(l) \tag{1}$$

The maximum average load supported by the bundle is

$$\bar{L}_{max} = \bar{L}(\hat{l}) = N\hat{l}R(\hat{l}) \tag{2}$$

where $\hat{l}$ satisfies the equation

$$\left.\left.\frac{d\bar{L}(l)}{dl}\right|_{l=\hat{l}} = 0 \quad \text{or} \quad R(\hat{l}) = \hat{l}b(\hat{l}) \right\} \tag{3}$$

Equations (2) and (3) represent the approximation to the breaking strength of the bundle as given by Pierce. Daniels showed that this result is the correct asymptotic limit for the expectation value of the breaking strength for large N. He also found that the asymptotic limit for the standard deviation of the strength for large N is given by

$$\sigma = \hat{l}\sqrt{NR(\hat{l})B(\hat{l})} \tag{4}$$

Daniels makes the pertinent remark that equation (4) would follow if one could assume that the breaking strength is only dependent on the number $\hat{n} = n(\hat{l})$ of fibers surviving at a load $\hat{l}$ where $\hat{n}$ is, of course, distributed according to the simple binomial law. He adds, however, that there appears to be no "a priori justification" for this assumption.

The average breaking strength $\bar{S}$ of a bundle of N fibers is the maximum load that the bundle supports averaged over the population of bundles. For a particular bundle, this maximum load will not, in general, occur at $l = \hat{l}$, and hence, it will be larger than $L(\hat{l})$, the value of L at $\hat{l}$. On the other hand, $\overline{L}_{max}$ is the $L(\hat{l})$ averaged over the population of bundles, and therefore, $\overline{L}_{max}$ constitutes a lower bound for $\overline{S}$. If the maximum load for every possible path (in the sense of figure 1(c)) occurred at $l = \hat{l}$, then $\overline{L}_{max} = \overline{L}(\hat{l})$ would indeed be $\overline{S}$ exactly. It can be said that the approximation to $\overline{S}$ as given by equation (2) treats every path as though its maximum load occurred at $l = \hat{l}$.

Average Path Between Two Designated Points

It shall first be shown that, for values of l lying between l_1 and l_2, the average path taken by bundles which are constrained to go through the points $R(l_1)$, n_1 and $R(l_2)$, n_2 (i.e., bundles which have n_1 unbroken fibers at a load l_1, and n_2 unbroken fibers at a load l_2) is merely the straight line connecting these two points on ~~the~~ a plot of n against $R(l)$. The proof is straightforward and goes as follows.

The probability that a bundle originally consisting of N fibers takes a path such that it has n_1 survivors at l_1 and n_2 survivors at l_2 where $N \geq n_1 \geq n_2 \geq 0$ and $0 \leq l_1 \leq l_2$ is

$$P(n_1 \text{ at } l_1, n_2 \text{ at } l_2) = \frac{N!}{(N - n_1)!(n_1 - n_2)!n_2!}$$

$$[B(l_1)]^{N-n_1}[R(l_1) - R(l_2)]^{n_1-n_2}[R(l_2)]^{n_2} \qquad (5)$$

The probability that a bundle originally consisting of N fibers takes a path that passes through the three points n_1 at l_1, n at l, and n_2 at l_2 where $N \geq n_1 \geq n \geq n_2 \geq 0$ and $0 \leq l_1 \leq l \leq l_2$, is

$$P(n_1 \text{ at } l_1, n \text{ at } l, n_2 \text{ at } l_2) = \frac{N!}{(N-n_1)!(n_1-n)!(n-n_2)!n_2!}$$

$$[B(l_1)]^{N-n_1}[R(l_1)-R(l)]^{n_1-n}[R(l)-R(l_2)]^{n-n_2}[R(l_2)]^{n_2} \qquad (6)$$

The conditional probability that a bundle which passes through n_1 at l_1 and n_2 at l_2 has n unbroken fibers at l is given by dividing equation (6) by equation (5)

$$P(n \text{ at } l/n_1 \text{ at } l_1, n_2 \text{ at } l_2) = \frac{(n_1-n_2)!}{(n_1-n)!(n-n_2)!}$$

$$\frac{[R(l_1)-R(l)]^{n_1-n}[R(l)-R(l_2)]^{n-n_2}}{[R(l_1)-R(l_2)]^{n_1-n_2}} \qquad (7)$$

The expectation value of n at l for a bundle passing through n_1 at l_1, and n_2 at l_2 is

$E(n \text{ at } l/n_1 \text{ at } l_1, n_2 \text{ at } l_2)$

$$= \sum_{n=n_2}^{n_1} nP(n \text{ at } l/n_1 \text{ at } l_1, n_2 \text{ at } l_2)$$

$$= n_2 + (n_1 - n_2)\frac{R(l)-R(l_2)}{R(l_1)-R(l_2)} \qquad (8)$$

where $N \geq n_1 \geq n \geq n_2 \geq 0$ and $0 \leq l_1 \leq l \leq l_2$.

Equation (8) shows that the average number of fibers surviving in a bundle passing through n_1 at l_1 and n_2 at l_2 is linear in $R(l)$. Therefore, the average path appears as a straight line connecting the points n_1 at l_1 and n_2 at l_2 on ~~the~~ a plot of n against $R(l)$.

Classification of Paths

In the calculations in this section, the paths will be grouped in accordance with $\hat{n}$, the number of fibers surviving at $\hat{l}$, and the average load for each class of paths shall be found as a function of l or $R(l)$. It shall be shown that the average load for paths 1 (fig. 2) having more fibers surviving at $\hat{l}$ than expected reaches a maximum value at $l = \hat{l}$ whereas the average load for paths 2 having fewer fibers surviving than expected at $\hat{l}$ reaches a maximum value at $l < \hat{l}$. This subdivision of paths in accordance with the value of $\hat{n}$ carries through at all values of l so that the average load $\bar{L}$ at any l for bundles having a given $\hat{n}$ may be determined. The subdivision can also be regarded as a first, although primitive step in the approach to the ideal wherein the probability and maximum supported load are determined for each possible path, so that an averaging of the maximum loads over the infinite population of paths yields $\bar{S}$, the average breaking strength of the bundle.

In accordance with equation (8), the expectation value of n at l for a bundle passing through $\hat{n}$ at $\hat{l}$ is

$$E(n \text{ at } l/\hat{n} \text{ at } \hat{l}) = \begin{cases} N - (N - \hat{n}) \dfrac{B(l)}{B(\hat{l})} & n \geq \hat{n}, l \leq \hat{l} \qquad (9a) \\ \hat{n} \dfrac{R(l)}{R(\hat{l})} & n \leq \hat{n}, l \geq \hat{l} \qquad (9b) \end{cases}$$

Figure 2 represents the situation that prevails. Every path must go through the initial point $n = N$, $R(l) = 1$ and the terminal point $n = 0$, $R(l) = 0$. If no other point along the path is specified, the path averaged over the bundles is given by curve 0 of figure 2, that is, by a straight line connecting the initial and terminal points. If a point is specified along the path between the initial and terminal points, two-segmented lines result that represent the average paths of bundles going through the additional point specified (curves 1 and 2 of fig. 2). The additional point specified for curve 1 is $\hat{n}_1$ at $\hat{l}$ whereas the additional point for curve 2 is $\hat{n}_2$ at $\hat{l}$.

When the load per fiber is l, the average total load supported by a bundle which goes through $\hat{n}$ at $\hat{l}$ is

$$\overline{L}(l/\hat{n} \text{ at } \hat{l}) = l\, E(n \text{ at } l/\hat{n} \text{ at } \hat{l}) \tag{10}$$

Maximum Loads on Paths

The maximum load that occurs on an average path when a particular $\hat{n}$ at $\hat{l}$ is specified can be determined as follows. Let $G(l) \equiv l\, R(l)$; then equations (1) and (3) can be written

$$\overline{L}(l) = N\, G(l) \tag{1a}$$

$$G'(\hat{l}) = 0 \tag{3a}$$

The assumption is also made that the form of $R(l)$ is such that $\overline{L}(l)$ exhibits one and only one peak, that peak occurring at $\hat{l}$. Mathematically this is equivalent to requiring that

$$G'(l) > 0 \qquad l < \hat{l} \tag{11a}$$

$$G'(l) < 0 \qquad l > \hat{l} \tag{11b}$$

From equations (9) and (10),

$$\frac{d\overline{L}(l/\hat{n} \text{ at } \hat{l})}{dl} = \begin{cases} \dfrac{\hat{n} - NR(\hat{l}) + (N - \hat{n})G'(l)}{B(\hat{l})} & l < \hat{l} \quad (12a) \\[2ex] \dfrac{\hat{n}}{R(\hat{l})}\, G'(l) & l > \hat{l} \quad (12b) \end{cases}$$

Equation (12a) shows that, for values of $\hat{n} \geq NR(\hat{l})$, $\bar{L}(l/\hat{n}$ at $\hat{l})$ is always increasing up to $l = \hat{l}$. At $l = \hat{l}$, the slope changes sign (eq. (12b)) and $\bar{L}(l/\hat{n}$ at $\hat{l})$ decreases as l increases when $l > \hat{l}$. Hence, when $\hat{n} > NR(\hat{l})$ the maximum value of $\bar{L}(l/\hat{n}$ at $\hat{l})$ occurs at $l = \hat{l}$ or

$$\bar{L}(l/\hat{n} \text{ at } \hat{l})_{max} = \hat{n}\hat{l} \qquad NR(\hat{l}) \leq \hat{n} \leq N \tag{13}$$

Again, by equations (12), for values of $\hat{n} \leq NR(\hat{l})$, $\bar{L}(l/\hat{n}$ at $\hat{l})$ reaches a maximum when $l = \lambda_{\hat{n}} \leq \hat{l}$, where $\lambda_{\hat{n}}$ satisfies the equation

$$G'(\lambda_{\hat{n}}) = \frac{NR(\hat{l}) - \hat{n}}{N - \hat{n}} \tag{14}$$

With $\delta_{\hat{n}}$ denoting the right side of equation (14) the following quantities can be expanded in powers of $\delta_{\hat{n}}$, where the coefficients of the powers of $\delta_{\hat{n}}$ are functions of the derivatives of $G(l)$ evaluated at $\hat{l}$:

$$\lambda_{\hat{n}} = \hat{l} + \frac{1}{G_2}\delta_{\hat{n}} - \frac{G_3}{2G_2^3}\delta_{\hat{n}}^2 + O\left(\delta_{\hat{n}}^3\right) \tag{15}$$

$$G(\lambda_{\hat{n}}) = G(\hat{l}) + \frac{1}{2G_2}\delta_{\hat{n}}^2 - \frac{1}{3}\frac{G_3}{G_2^3}\delta_{\hat{n}}^3 + O\left(\delta_{\hat{n}}^4\right) \tag{16}$$

$$\bar{L}(\lambda_{\hat{n}}/\hat{n} \text{ at } \hat{l}) = \hat{n}\,\hat{l} - \frac{N}{2G_2}\delta_{\hat{n}}^2 + \frac{N}{2}\left(\frac{G_3}{3G_2^3} - \frac{1}{G_2}\right)\delta_{\hat{n}}^3 + O\left(\delta_{\hat{n}}^4\right) \tag{17}$$

In equations (15) to (17) the symbol G_n represents the n^{th} derivative of $G(l)$ with respect to l evaluated at $l = \hat{l}$. Note that $G_2 < 0$.

Equations (13) and (17) show that to first order in $\delta_{\hat{n}}$, when the paths are classified in accordance with $\hat{n}$ at $\hat{l}$, Daniels' result holds, that is, the breaking strength of the bundle is only dependent on the number of survivors at $\hat{l}$.

Equation (17) may be employed to find a larger lower bound for $\overline{S}$, the average breaking strength, than that given by equation (2):

$$\overline{S} > \sum_{\hat{n}=0}^{N} P(\hat{n} \text{ at } \hat{l})\overline{L}(\lambda_{\hat{n}}/\hat{n} \text{ at } \hat{l})$$

$$> N\hat{l}R(\hat{l}) + \frac{N}{2} \sum_{\hat{n}=0}^{[NR(\hat{l})]} P(\hat{n} \text{ at } \hat{l}) \left[-\frac{1}{G_2}\delta_{\hat{n}}^2 + \left(\frac{1}{3}\frac{G_3}{G_2^3} - \frac{1}{G_2}\right)\delta_{\hat{n}}^3 + \ldots \right] \tag{18}$$

If the variable is changed from $\hat{n}$ to y, where y is of standard measure and is defined by

$$y = \frac{NR(\hat{l}) - \hat{n}}{\sqrt{NR(\hat{l})B(\hat{l})}} \tag{19}$$

the frequency function $f(y)$ can be expanded in terms of Hermite polynomials $H_n(y)$ by the method of Kendall and Stuart (ref. 3):

$$f(y) \cong \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left[1 + \frac{1}{6}\mu'_{3:y} H_3(y) + \ldots\right]$$

$$\cong \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \left[1 + \frac{1}{6}\frac{2R(\hat{l}) - 1}{\sqrt{NR(\hat{l})B(\hat{l})}}(y^3 - 3y) + \ldots\right] \tag{20}$$

Also $\delta_{\hat{n}}$ can be written as a series in powers of y:

$$\delta_{\hat{n}} = \sqrt{\frac{R(\hat{l})}{NB(\hat{l})}}\, y \sum_{k=0}^{\infty} (-1)^k \left[\frac{R(\hat{l})}{NB(\hat{l})}\right]^{k/2} y^k \tag{21}$$

Substituting in equation (18) yields

$$\bar{S} > N\hat{l}R(\hat{l}) + \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\left[1 + \frac{1}{6}\frac{2R(\hat{l}) - 1}{\sqrt{NR(\hat{l})B(\hat{l})}}(y^3 - 3y) + \cdots\right]$$

$$\left\{-\frac{1}{G_2}\frac{R(\hat{l})}{B(\hat{l})}y^2 + \left(\frac{G_3}{3G_2^3} + \frac{1}{G_2}\right)\left[\frac{R(\hat{l})}{B(\hat{l})}\right]^{3/2}\frac{y^3}{\sqrt{N}} + \cdots\right\}dy$$

$$> N\hat{l}R(\hat{l}) - \frac{1}{4G_2}\frac{R(\hat{l})}{B(\hat{l})}\left[1 + \frac{2}{3}\frac{2R(\hat{l}) - 1}{\sqrt{2\pi NR(\hat{l})B(\hat{l})}} + \cdots\right]$$

$$+ \left(\frac{G_3}{3G_2^3} + \frac{1}{G_2}\right)\frac{1}{\sqrt{2\pi N}}\left[\frac{R(\hat{l})}{B(\hat{l})}\right]^{3/2}\left[1 + \cdots\right] + O(N^{-1}) \qquad (22)$$

The variance of the breaking load based on the previous approximations is given as

$$\sigma^2 = NR(\hat{l})B(\hat{l})(\hat{l})^2 + \sqrt{\frac{2}{\pi}}\frac{\hat{l}}{G_2}\sqrt{\frac{NR(\hat{l})}{B(\hat{l})}}R(\hat{l})$$

$$\left[1 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{2R(\hat{l}) - 1}{\sqrt{NR(\hat{l})B(\hat{l})}} + \cdots\right] + O(1) \qquad (23)$$

CONCLUDING REMARKS

A method of determining a lower bound for the average breaking strength of a bundle of classical fibers has been presented which involves the subdivision of the bundles in accordance with the number of unbroken fibers at a given value of load per fiber. The method also yields the approximate statistical distribution of the breaking strengths so that the variance of strengths may be calculated. It has been found that the average bundle of every subdivision breaks at about the same value of the load per fiber.

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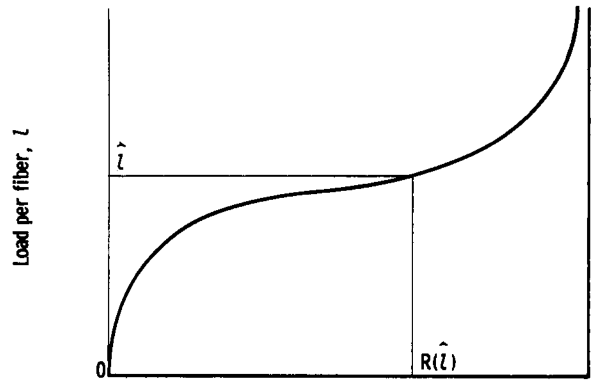
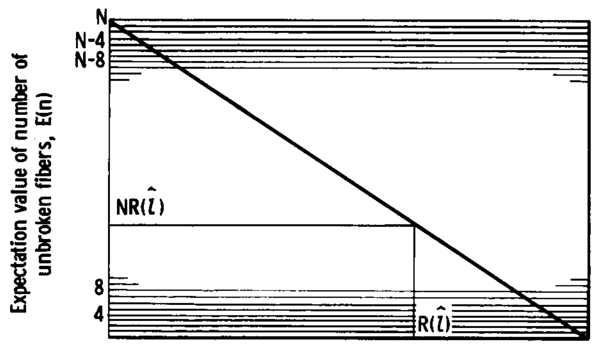
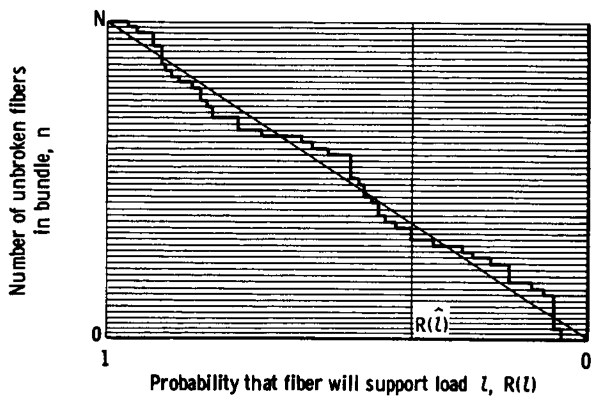


(a) Load.



(b) Expectation of survival.



(c) Number of unbroken fibers.

Figure 1. - Plots with $R(l)$ as independent variable.

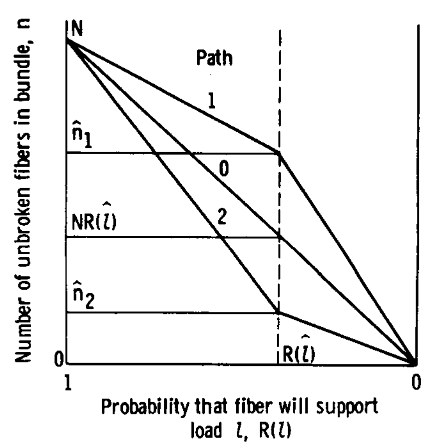


Figure 2. - Average paths of bundles.